TRUNCATION OF MULTILINEAR HANKEL OPERATORS

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ABSTRACT. We extend to multilinear Hankel operators the fact that truncation of bounded Hankel operators is bounded. We prove and use a continuity property of a kind of bilinear Hilbert transforms on product of Lipschitz spaces and Hardy spaces.

1. Statement of the results

In this note, we prove that truncations of bounded multilinear Hankel operators are bounded. This extends the same property for linear Hankel operators, a result obtained by [BB], which we first recall. A matrix $B = (b_{mn})_{m,n\in\mathbb{N}}$ is called of *Hankel type* if $b_{mn} = b_{m+n}$ for some sequence $b \in l^2(\mathbb{N})$. We can identify B with an operator acting on $l^2(\mathbb{N})$. Moreover, if we identify $l^2(\mathbb{N})$ with the complex Hardy space $\mathcal{H}^2(\mathbb{D})$ of the unit disc, then B can be realized as the integral operator, called *Hankel operator* and denoted by H_b, which acts on $f \in \mathcal{H}^2(\mathbb{D})$ by

$$H_b f(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{b(\zeta)f(\overline{\zeta})}{1-\overline{\zeta}z} d\sigma(\zeta).$$

In other words, $H_b f = \mathcal{C}(b\check{f})$ where $\mathcal{C}$ denotes the Cauchy integral, $\check{f}(\zeta) := f(\overline{\zeta})$, $\zeta \in \mathbb{T}$. The *symbol* b of the Hankel operator is given by $b(\zeta) := \sum_{k=0}^{\infty} b_k\zeta^k$. If $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$, one has

$$H_b f(z) = \sum_{m\in\mathbb{N}} (\sum_{n\in\mathbb{N}} a_n b_{m+n}) z^m.$$

Now, we consider truncations of matrices defined as follows. For $\beta, \gamma \in \mathbb{R}$, the truncated matrix $\Pi_{\beta,\gamma}(B)$ is the matrix whose (m,n) entry is b_{mn} or zero, depending on the fact that $m \geq \beta n + \gamma$ or not. It is proved in [BB] that such truncations, for $\beta \neq -1$, preserve the boundedness for Hankel operators. The proof consists in showing that truncations are closely related to bilinear periodic Hilbert transforms. One then uses the theorem of Lacey-Thiele (see [LT1], [LT2], [LT3]) in the periodic

1991 *Mathematics Subject Classification.* 47B35(42A50 47A63 47B10 47B49).

Key words and phrases. Hankel operator, truncation, Hardy spaces, Lipschitz spaces, bilinear Hilbert transform.

Authors partially supported by the 2002-2006 IHP Network, Contract Number: HPRN-CT-2002-00273 - HARP

The authors would like to thank Joaquim Bruna who suggested this problem.

setting. We are interested in the same problem for multilinear Hankel operators. For $n \in \mathbb{N}$, we define the multilinear Hankel operator $H_b^{(n)}$ as follows. Let $f_1, \dots, f_n \in \mathcal{H}^2(\mathbb{D})$,

$$\begin{aligned} H_b^{(n)}(f_1, \dots, f_n)(z) &= \frac{1}{2\pi} \int_{\mathbb{T}} \frac{b(\zeta) f_1(\overline{\zeta}) \dots f_n(\overline{\zeta})}{1 - \overline{\zeta} z} d\sigma(\zeta) \\ &= H_b(f_1 \times \dots \times f_n)(z). \end{aligned}$$

When equipped with the canonical basis of $\mathcal{H}^2(\mathbb{D})$, this operator has a matrix B with entries in $\mathbb{N}^{n+1}$, which we denote by $B = (b_{i_0,\dots,i_n})_{i_0,\dots,i_n \in \mathbb{N}}$. We speak of $(n+1)$-dimensional infinite matrices (so that a usual matrix is a 2-dimensional matrix in our terminology). Its action on n vectors $a^1 \dots, a^n$ gives the vector whose m-th coordinate is

$$\sum_{i_1,\dots,i_n} b_{m,i_1,\dots,i_n} a^1_{i_1} \dots a^n_{i_n}.$$

In the case of the operator $H_b^{(n)}$, the matrix B is a $(n+1)$-dimensional matrix with entries which are constant on the hyperplanes $i_0 + \dots + i_n = c$. Such a matrix is called a $(n+1)$-dimensional Hankel matrix.

We consider truncations of $(n+1)$- dimensional matrices as follows. For $\beta \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, $\beta = (\beta_1, \dots, \beta_n)$, $\Pi_{\beta,\gamma}(H_b^{(n)})$ denotes the $(n+1)$-dimensional matrix whose $(i_0, \dots, i_n)$ entry is $b_{i_0+\dots+i_n}$ if $\beta_1 i_1 + \dots + \beta_n i_n + \gamma \leq i_0$ and zero otherwise. In this note, we consider the simplest case where $\beta = (1, \dots, 1)$ and $\gamma = 0$ which is denoted by $\Pi_{1,0}$. We will study the general case in a foregoing paper. Our main result is the following.

THEOREM 1. *If $H_b^{(n)}$ is a bounded multilinear Hankel operator from $(\mathcal{H}^2(\mathbb{D}))^n$ into $\mathcal{H}^2(\mathbb{D})$ then so is its truncated operators $\Pi_{1,0}(H_b^{(n)})$.*

Theorem 1 is a corollary of an estimate on a kind of bilinear Hilbert transform in the periodic setting which is of independent interest. Let us first give some notations. The usual Lipschitz spaces of order α are denoted by $\Lambda_\alpha(\mathbb{T})$, while $\mathcal{H}^p(\mathbb{T})$ denotes the real Hardy space, $p > 0$.

Let us finally recall that for f and b trigonometric polynomials on the torus, the periodic bilinear Hilbert transform of f and b is given by

$$\mathcal{H}(b, f)(x) = p.v. \int_{\mathbb{T}} b(x+t) f(2t) \frac{dt}{\tan \frac{x-t}{2}}.$$

Lacey-Thiele's Theorem, once transferred to the periodic setting, is the following.

THEOREM 2. [BB] *Let $1 < p, q \leq \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{3}{2}$. Then, there exists a constant $C > 0$ so that, for any trigonometric polynomials f and b,*

$$\|\mathcal{H}(b, f)\|_r \leq C \|f\|_p \|b\|_q.$$

We adapt the definition to our setting, and define

$$\widetilde{\mathcal{H}}(b,f)(x)=\int_{\mathbb{T}}\left(b(x+t)-b(2x)\right)f(2t)\frac{dt}{\tan\frac{x-t}{2}}. \tag{1}$$

We prove the following.

THEOREM 3. *Let $1<p<\infty$, $0<q<p$ and $\alpha=\frac{1}{q}-\frac{1}{p}$. There exists a constant $C>0$ so that, for any sufficiently smooth functions $b\in\Lambda_\alpha(\mathbb{T})$ and $f\in\mathcal{H}^p(\mathbb{T})$*

$$\|\widetilde{\mathcal{H}}(b,f)\|_{\mathcal{H}^p}\leq C\|f\|_{\mathcal{H}^q}\|b\|_{\Lambda_\alpha}. \tag{2}$$

We remark that the limiting case $b\in L^\infty(\mathbb{T})$ is given by the Lacey-Thiele theorem, Theorem 2.

Let us come back to holomorphic functions and to truncations. Denote by $\Lambda_\alpha(\mathbb{D})$, $\alpha>0$, the space of functions which are holomorphic in $\mathbb{D}$ and whose boundary values are in $\Lambda_\alpha(\mathbb{T})$. Denote also by $\mathcal{H}^p(\mathbb{D})$ the complex Hardy space on the unit disc, $p>0$. Recall that the dual of $\mathcal{H}^p(\mathbb{D})$ is $\Lambda_\alpha(\mathbb{D})$, for $p=\dfrac{1}{\alpha+1}$ ([D]). As an easy consequence of duality and factorization, one obtains that the Hankel operator H_b is bounded from $\mathcal{H}^q(\mathbb{D})$ into $\mathcal{H}^p(\mathbb{D})$, with $q<p$ and $p>1$, if and only if the symbol b is in $\Lambda_\alpha(\mathbb{D})$ with $\alpha=\frac{1}{q}-\frac{1}{p}$. More precisely, there exists a constant C such that, for all holomorphic polynomials f,

$$\|H_b(f)\|_p\leq C\|b\|_{\Lambda_\alpha(\mathbb{D})}\times\|f\|_{\mathcal{H}^q(\mathbb{D})}. \tag{3}$$

Theorem 3 has the following corollary, which gives the link with truncations.

COROLLARY 4. *Let $1<p<\infty$ and $0<q<p$. Let $b\in\Lambda_\alpha(\mathbb{D})$. Then the operator $\Pi_{1,0}(H_b)$ is bounded from $\mathcal{H}^q(\mathbb{D})$ into $\mathcal{H}^p(\mathbb{D})$.*

So, if H_b is a bounded operator, its truncate $\Pi_{1,0}(H_b)$ is also bounded.

Let us deduce Theorem 1 from the corollary. It is clear that

$$H_b^{(n)}(f_1,\ldots,f_n)(z)=H_b(f_1\times\cdots\times f_n)(z).$$

Using the factorisation of functions in Hardy classes, we know that $H_b^{(n)}$ is bounded as an operator from $\left(\mathcal{H}^2(\mathbb{D})\right)^n$ into $\mathcal{H}^2(\mathbb{D})$ if and only if the Hankel operator H_b is bounded from $\mathcal{H}^{2/n}(\mathbb{D})$ into $\mathcal{H}^2(\mathbb{D})$, that is, if and only if b is in $\Lambda_\alpha(\mathbb{D})$ for $\alpha=\frac{n-1}{2}$. To conclude, we use the fact that the truncation $\Pi_{1,0}$ of $H_b^{(n)}$ corresponds to the truncation $\Pi_{1,0}$ of H_b, as it can be easily verified.

The remainder of the paper is organized as follows. In the next section, we deduce the corollary from Theorem 3. In the last one, we prove Theorem3. Let us emphasize the fact that this last proof does not use Lacey-Thiele Theorem, and is elementary compared to it.

2. The link between truncations and bilinear Hilbert transforms

We prove the corollary. It is sufficient to prove that, for b and f trigonometric polynomials,

$$\|\Pi_{1,0}H_b(f)\|_p \leq C\|b\|_{\Lambda_\alpha(\mathbb{D})} \times \|f\|_{\mathcal{H}^q(\mathbb{D})} \tag{4}$$

for some constant C which is independent of b and f.

Let b and f be two triginometric polynomials. Assume that $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$ and denote by F the function defined on the torus by $F(x) = f(e^{-ix})$. It is elementary to see that F and f have the same norm in $\mathcal{H}^q(\mathbb{T})$. Moreover, an elementary computation (which is already in [BB]) shows that the analytic part of $\mathcal{H}(b, F)(x)$ is equal to

$$\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} a_n b_{m+n} sign((m-n))e^{i2mx}.$$

So it is sufficient to prove that $\mathcal{C}(\mathcal{H}(b, F))$ satisfies the desired estimate,

$$\|\mathcal{C}(\mathcal{H}(b, F))\|_p \leq C\|b\|_{\Lambda_\alpha(\mathbb{D})} \times \|f\|_{\mathcal{H}^q(\mathbb{D})}. \tag{5}$$

We want to replace $\mathcal{H}(b, F)$ by $\widetilde{\mathcal{H}}(b, F)$, for which we have such an estimate given in Theorem 3. Let us look at the difference, which is given, up to a constant, by the Cauchy projection of

$$x \mapsto b(2x)\int_{\mathbb{T}} F(2t)\frac{dt}{\tan\frac{x-t}{2}} = b(2x)F(2x)$$

since f has only non zero coefficients for positive frequencies. We recognize $H_b(f)(z^2)$, whose norm in $\mathcal{H}^p(\mathbb{D})$ coincides with the one of $H_b(f)$. To conclude for (5), we use (2) and (3).

3. Proof of the Theorem 3.

When $q > 1$, then the kernel of $\widetilde{\mathcal{H}}$ is bounded , up to a constant $c\|b\|_\alpha$, by the Riesz potential $|x-y|^{-1+\alpha}$, and the estimate follows directly. Let us now concentrate on $q \leq 1$, for which we can use the atomic decomposition. By the atomic decomposition Theorem of $H^q(\mathbb{T})$, it suffices to consider the action of $\widetilde{\mathcal{H}}(b, .)$ on $H^q(\mathbb{T})$-atoms. Let a be a $H^q(\mathbb{T})$-atom.

If a is the constant atom, or if a is a non constant atom which is supported in some interval I of the torus of length bigger than $\frac{\pi}{4}$, then, for all $r > 1$, its L^r norm is uniformly bounded. It follows at once that the L^p norm of $\widetilde{\mathcal{H}}(b, a)$ is also uniformly bounded.

We assume now that $a : t \mapsto \tilde{a}(2t)$ is an atom supported in some interval I on the torus, centered at x_I and of radius $r < \frac{\pi}{4}$. Denote by $\tilde{I}$ the interval centered at x_I and of radius $2r$, and by $2\tilde{I}$ the ball centered

at $2x_I$ and of radius $4r$. We first consider the case when $0<\alpha<1$. We write

$$\begin{aligned}\widetilde{\mathcal{H}}(b,\tilde a)(x) &= \underbrace{p.v.\int_{\mathbb{T}}(b(x+t)-b(2x))a(t)\frac{dt}{\tan\frac{x-t}{2}}1\!\!1_{x\in\tilde I}}_{A_1(x)}\\ &+ \underbrace{p.v.\int_{\mathbb{T}}(b(x+t)-b(2x))a(t)\frac{dt}{\tan\frac{x-t}{2}}1\!\!1_{x\in\tilde I^c}}_{A_2(x)}\end{aligned}$$

We prove that both A_1 et A_2 are L^p-functions. To prove that $A_1\in L^p$, we write $|A_1(x)|\le\|b\|_\alpha\mathcal{I}_\alpha(|a|)(x)1\!\!1_{x\in\tilde I}$ where $\mathcal{I}_\alpha$ denotes the fractional integral related to the Riesz potential $|x-y|^{-1+\alpha}$. So, by Minkowski inequality,

$$\|A_1\|_{L^p}\le c\|b\|_\alpha\|\mathcal{I}_\alpha(a)||_{L^s}\times|I|^{1/p-1/s}$$

for any $s>p$. We choose s large enough to have $\alpha+1/s<1$ and $r>1$ so that $1/r=\alpha+1/s$. For these choices, we get

$$\|A_1\|_{L^p}\le c\|b\|_\alpha\|a\|_{L^r}\times|I|^{1/p-1/s}\le c\|b\|_\alpha.$$

It remains to consider the term denoted by A_2. For this term, we write

$$b(x+t)-b(2x)=[b(x+t)-b(x+x_I)]+[b(x+x_I)-b(2x)].$$

The corresponding terms are denoted by $A_2^{(1)}$ and $A_2^{(2)}$ respectively. For the first term $A_2^{(1)}$, we use that

$$|b(x+t)-b(x+x_I)|\le c\|b\|_\alpha|I|^\alpha$$

for any $t\in I$ and that $|\tan\frac{x-t}{2}|\ge C|x-x_I|$ when $t\in I$ and $x\in\tilde I^c$. So,

$$\begin{aligned}|A_2^{(1)}(x)| &\le c\|b\|_\alpha|I|^\alpha\times\frac{1}{|x-x_I|}\left(\int_I|a(t)|dt\right)1\!\!1_{\tilde I^c}(x)\\ &\le c\|b\|_\alpha|I|^{\alpha+1-1/q}\frac{1}{|x-x_I|}1\!\!1_{\tilde I^c}(x).\end{aligned}$$

Taking the L^p-norm, it gives, as $p>1$,

$$\|A_2^{(1)}\|_{L^p}\le c\|b\|_\alpha|I|^{\alpha+1-1/q}\left(\int_{|x-x_I|\ge 2r}\frac{1}{|x-x_I|^p}dx\right)^{1/p}\le c\|b\|_\alpha.$$

For the second part, we use the fact that a has vanishing moment of order $m:=\left[\frac{1}{q}\right]-1$ so that one can substract to $t\to\frac{1}{\tan\frac{x-t}{2}}$ its Taylor expansion of order m at point x_I without changing the value of $A_2^{(2)}$. As the corresponding difference is bounded by $\dfrac{|I|^{m+1}}{|x-x_I|^{m+2}}$ for $t\in I$

and $x \in \tilde{I}^c$, it allows to obtain

$$\begin{aligned} |A_2^{(2)}(x)| &\leq c\|b\|_\alpha |x-x_I|^\alpha \times \frac{|I|^{m+1}}{|x-x_I|^{m+2}} \times \left(\int_{\tilde{I}} |a|\right) 1\!\!1_{\tilde{I}^c}(x) \\ &\leq c\|b\|_\alpha |x-x_I|^{\alpha-m-2} \times |I|^{m+2-1/q} 1\!\!1_{\tilde{I}^c}(x). \end{aligned}$$

Eventually, it gives

$$\|A_2^{(2)}\|_p \leq c\|b\|_\alpha |I|^{m+2-1/q} \times \left(\int_{|x-x_I|\geq 2r} |x-x_I|^{(\alpha-m-2)p} dx\right)^{1/p}.$$

This last integral is convergent since, as $m = \left[\frac{1}{q}\right] - 1$ and $\alpha = \frac{1}{q} - \frac{1}{p}$, $(\alpha - m - 2)p = -1 + (\frac{1}{q} - m - 2)p < -1$. So, we obtain

$$\|A_2^{(2)}\|_p \leq c\|b\|_\alpha.$$

So, we have proved that $\|\widetilde{\mathcal{H}}(b,a)\|_{L^p} \leq c\|b\|_\alpha$ for any H^q-atom a. It proves that $\widetilde{\mathcal{H}}(b,\cdot)$ maps $H^q(\mathbb{T})$ into $L^p(\mathbb{T})$ boundedly. It ends the proof of the theorem in the case $0 < \alpha < 1$.

Now, we illustrate the method for larger values of α by considering the case $1 \leq \alpha < 2$. We write

$$\begin{aligned} \tilde{\mathcal{H}}(b,a)(x) &= \int_{\mathbb{T}} (b(x+t) - b(2x) - (x-t)b'(2x))a(t)\frac{dt}{\tan\frac{x-t}{2}} \\ &+ b'(2x)K * a(x) \\ &= H_1(x) + H_2(x). \end{aligned}$$

Here K is the $\mathcal{C}^\infty$-kernel defined by $K(x) := \frac{x}{\tan\frac{x}{2}}$, $x \in \mathbb{T}$. The corresponding term H_2 is hence in $H^p(\mathbb{T})$ since, as K is a $\mathcal{C}^\infty$-kernel, $K * a$ is a smooth function (even if a is only a distribution in $H^q(\mathbb{T})$). In particular it belongs to $L^p(\mathbb{T})$ and so is for $b'(2.)K * a$ with

$$\|b'(2.)K * a\|_{H^p} \leq C\|b'\|_\infty \leq C\|b\|_\alpha.$$

So, the problem reduces to show that H_1 belongs to $L^p(\mathbb{T})$. We write as before $H_1(x) = A_1(x) + A_2(x)$ where $A_1(x) = H_1(x)1\!\!1_{\tilde{I}}(x)$.

To prove that $A_1 \in L^p(\mathbb{T})$, we write, for $x \in \tilde{I}$

$$\begin{aligned} (*) &:= b(x+t) - b(2x) - (t-x)b'(2x) \\ &= [b(x+t) - b(2x_I) - (x+t-2x_I)b'(2x_I)] \\ &- [b(2x) - b(2x_I) - (2x-2x_I)b'(2x_I)] \\ &+ (t-x)[b'(2x_I) - b'(2x)] \\ &:= \tilde{b}(x+t) - \tilde{b}(2x) + (t-x)[b'(2x_I) - b'(2x)] \end{aligned}$$

where $\tilde{b}(s) := [b(s) - b(2x_I) - (s-2x_I)b'(2x_I)]\Psi(s)$ with Ψ a smooth function supported in twice of $2\tilde{I}$ identically 1 in $2\tilde{I}$. We write $A_1 = A_1^{(1)} + A_1^{(2)}$. For the first term, we remark that $\tilde{b}$ belongs to $\Lambda_\beta(\mathbb{T})$ for any $0 < \beta < 1$ with

$$\|\tilde{b}\|_{\Lambda_\beta} \leq c\|b\|_\alpha |I|^{\alpha-\beta}.$$

This follows from the fact that $\|\tilde{b}'\|_\infty \leq c\|b\|_\alpha |I|^{\alpha-1}$ (since, by the choice of Ψ, $\|\Psi'\|_\infty \leq c|I|^{-1}$). From the first part of the proof, we get that, for $\beta = 1 - \dfrac{1}{p}$, the corresponding operator maps $H^1(\mathbb{T})$ into $H^p(\mathbb{T})$ with

$$\|A_1^{(1)}\|_{L^p} \leq c\|\tilde{b}\|_{\Lambda_\beta}\|a\|_{L^1} \leq c|I|^{\alpha-\beta+1-1/q} \leq c.$$

For $A_1^{(2)}$, we have $A_1^{(2)}(x) = [b'(2x_I) - b'(2x)]K * a(x)1\!\!1_{\tilde{I}}(x)$. So, it gives

$$\|A_1^{(2)}\|_{L^p} \leq c\|b\|_\alpha |I|^{\alpha-1+1/p} \times \int_{\mathbb{T}} |a| \leq c\|b\|_\alpha.$$

To deal with A_2, we write

$$\begin{array}{rcl} (*) & = & b(x+t) - b(2x) - (t-x)b'(2x) \\ & = & [b(x+t) - b(x+x_I) - (t-x_I)b'(x+x_I)] \\ & + & [b(x+x_I) - b(2x) - (x_I - x)b'(x+x_I)] \\ & + & (t-x)[b'(x+x_I) - b'(2x)]. \end{array}$$

The corresponding terms are denoted by $A_2^{(1)}$, $A_2^{(2)}$ and $A_2^{(3)}$ respectively. For the first term $A_2^{(1)}$, we use that $|b(x+t) - b(x+x_I) - (t - x_I)b'(x+x_I)| \leq c\|b\|_\alpha |I|^\alpha$ for any $t \in I$ and that $|\tan\frac{x-t}{2}| \geq C|x - x_I|$ when $t \in I$ and $x \in \tilde{I}^c$. So, the estimate of this term is as before. For the second part, we use the fact that a has vanishing moment of order $m := \left[\frac{1}{q}\right] - 1$ so that one can substract to $t \to \frac{1}{\tan\frac{x-t}{2}}$ its Taylor expansion of order m at point x_I without changing the value of $A_2^{(2)}$. As the corresponding difference is bounded by $\dfrac{|I|^{m+1}}{|x - x_I|^{m+2}}$ for $t \in I$ and $x \in \tilde{I}^c$, it allows to obtain the same estimate as before.

We just have to consider the third term $A_2^{(3)}$. Here, we write that $t - x = [t - x_I] + [x_I - x]$ so that it gives two different terms to estimate. In the first, we use again that a has vanishing moment of order less than m so that one can substract to $t \to \frac{1}{\tan\frac{x-t}{2}}$ its Taylor expansion of order $m-1$ at point x_I without changing the value of the integral $\displaystyle\int_{\mathbb{T}} [t - x_I]a(t)\frac{dt}{\tan\frac{x-t}{2}}$. So, it gives that the corresponding term is bounded by $\|b\|_\alpha |x - x_I|^{\alpha-1-m-1}|I|^{2-1/q+m}$ and its corresponding L^p-norm is bounded by $C\|b\|_\alpha$. For the very last term, one can substract to $t \to \frac{1}{\tan\frac{x-t}{2}}$ its Taylor expansion of order m at point x_I without changing the value of the integral $\displaystyle\int_{\mathbb{T}} a(t)\frac{dt}{\tan\frac{x-t}{2}}$, it gives that the corresponding term has the same bound as the preceding one. It finishes the proof.

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